INFLUENCE OF STRONG SELF ELECTRIC FIELDS ON THE ION RESONANCE INSTABILITY

IN A NONNEUTRAL PLASMA COLUMN

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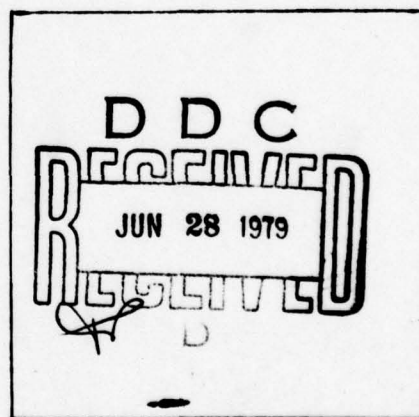
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INFLUENCE OF STRONG SELF ELECTRIC FIELDS ON THE ION RESONANCE INSTABILITY

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The influence of strong self-electric fields on the ion resonance
instability is examined for a cylindrical nonneutral plasma column
immersed in a uniform axial magnetic field $B_0 \hat{e}_z$. The analysis is
carried out within the framework of a macroscopic cold-fluid model,
and electrostatic stability properties are investigated for the case
of rectangular electron and ion density profiles. The parameter
$\delta = (2\hat{\omega}_{pe}^2/\omega_{ce}^2)(1-f)$ is introduced as a convenient measure of the relative
strengths of the equilibrium self-electric force and the magnetic force
on an electron fluid element. (Here, $\hat{\omega}_{pe}$ is the electron plasma
frequency, ω_{ce} is the electron cyclotron frequency, $f = n_i^0/n_e^0$ is the
fractional charge neutralization, and $\delta = 1$ corresponds to the maximum
allowed charge density for radial confinement of the equilibrium
configuration.) An important conclusion of this study is that the
equilibrium self-electric field can have a large influence on stability
behavior. In particular, stability properties for $\delta \lesssim 1$ differ
substantially from those obtained when $\delta \ll 1$. Moreover, for a
nonneutral plasma column with significant charge neutralization,
it is found that the fundamental mode ($\ell = 1$) is <u>not</u> the most unstable
mode. Rather, higher harmonic perturbations have larger growth rates.
However, in the limiting case where $f \ll 1$ and $\delta \ll 1$, the $\ell = 1$ mode can
have the largest growth rate, which is consistent with the result
previously obtained by Levy et al.

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1. INTRODUCTION

There have been several recent experimental and theoretical studies of the fundamental equilibrium and stability properties of nonneutral plasmas[1] in both mirror[2-5] and uniform[6-10] magnetic field configurations. Perhaps one of the most basic instabilities that characterizes a nonneutral plasma with both electron and ion components is the ion resonance instability.[11-13] In cylindrical geometry (Fig. 1), the ion resonance instability can be described as a two-rotating-stream instability[11] in which the relative rotation between electrons and ions is produced by the equilibrium self-electric field $E_r^0(r)\hat{e}_r$. Previous analyses[11,12] of this instability have been restricted to low beam densities and small values of fractional charge neutralization, i.e.,

$$\frac{2\hat{\omega}_{pe}^2}{\omega_{ce}^2}(1-f) \ll 1 ,$$
$$f \ll 1 .$$

Here, $\hat{\omega}_{pe}$ is the electron plasma frequency, ω_{ce} is the electron cyclotron frequency, and $f = n_i^0/n_e^0$ is the fractional charge neutralization provided by the positive ion background. In this paper, we investigate the influence of strong self-electric fields on the ion resonance instability in a nonneutral plasma column with rectangular electron and ion density profiles (Fig. 2). The analysis is carried out within the framework of a macroscopic model, treating the electrons and ions as cold ($T_j \to 0$) fluids immersed in a uniform axial magnetic field $B_0\hat{e}_z$. Moreover, the stability studies assume electrostatic perturbations with infinitely long axial wavelength ($\partial/\partial z = 0$).

It is useful to introduce the dimensionless parameter

$$\delta \equiv \frac{2\hat{\omega}_{pe}^2}{\omega_{ce}^2}(1-f) \, ,$$

which is a measure of the characteristic relative strength of the equilibrium self-electric force and the magnetic force on an electron fluid element. In the present equilibrium and stability analysis, the parameter δ and the fractional charge neutralization f are allowed to span the range of values

$$0 < \delta \leq 1 \, ,$$

and

$$0 \leq f \leq 1 \, ,$$

where $\delta = 1$ corresponds to the maximum allowed charge density for radial confinement of the equilibrium configuration. One of the most important conclusions of this study is that the equilibrium self-electric field can have a large influence on stability behavior. In particular, stability properties for $\delta \lesssim 1$ differ substantially from those obtained when $\delta \ll 1$ (see, for example, Fig. 5). Moreover, for a nonneutral plasma column with significant charge neutralization, it is found that the fundamental mode ($\ell=1$) is not the most unstable mode. Rather, higher harmonic perturbations have larger growth rates (see, for example, Figs. 8 and 9). However, in the special limiting case where $f \ll 1$ and $\delta \ll 1$, the $\ell=1$ mode can have the largest growth rate, which is consistent with the result previously obtained by Levy et al.[12] [Fig. 8].

The outline of this paper is the following. In Section 2, we give a brief description of the theoretical model (Section 2.A) and summarize the equilibrium properties for general density profiles $n_j^0(r)$, $j=e,i$ (Section 2.B). The corresponding eigenvalue equation

that describes the electrostatic stability properties of the nonneutral plasma column is derived in Section 3.A [Eq. (13)]. In Sections 3.B and 3.C, the eigenvalue equation is solved for the case where the electron and ion density profiles are rectangular [Eqs. (20)-(22)]. This leads to a fourth-order algebraic dispersion relation that determines the complex eigenfrequency ω [Eq. (26)]. A detailed numerical analysis of the dispersion relation is presented in Section 4, where stability properties are investigated for a broad range of plasma parameters $\hat{\omega}_{pe}^2/\omega_{ce}^2$, f, etc.

Finally, we emphasize that the present analysis is based on a macroscopic cold-fluid model in which the ion (and electron) motion is assumed to be laminar. Although this is a reasonable approximation when $r_{Li} \ll R_p$ (where r_{Li} is the characteristic thermal ion Larmor radius, and R_p is the radius of the plasma column), we expect significant modifications to the stability behavior when $r_{Li} \sim R_p$. The influence of finite ion Larmor radius effects on the ion resonance instability is currently under investigation[14] within the framework of a hybrid Vlasov-fluid model, which treats the electrons as a macroscopic, cold fluid and the ions in a fully kinetic (Vlasov) manner.

2.   THEORETICAL MODEL AND EQUILIBRIUM PROPERTIES

2.A   Theoretical Model

As illustrated in Fig. 1, the equilibrium configuration consists
of a cylindrically symmetric nonneutral plasma column that is infinite
in axial extent and aligned parallel to a uniform applied magnetic field
$B_0\hat{e}_z$. The mean motion of the plasma components is in the azimuthal
direction, and the applied magnetic field provides radial confinement
of the plasma column. As shown in Fig. 1, the characteristic radius
of the plasma column is denoted by R_p, and we introduce a cylindrical
polar coordinate system (r,θ,z) with the z axis coinciding with the axis
of symmetry; r is the radial distance from the z axis, and θ is the polar
angle in a plane perpendicular to the z axis. The deviation from equil-
ibrium charge neutrality $[n_e^0(r) \neq n_i^0(r)]$ produces an equilibrium
radial electric field $\underset{\sim}{E}^0(\underset{\sim}{x}) = E_r^0(r)\hat{e}_r$ that influences the azimuthal motion
of the plasma components.

The following are the main assumptions pertaining to the equilibrium
and stability analysis:

(a) Equilibrium properties are independent of z $(\partial/\partial z=0)$ and
azimuthally symmetric $(\partial/\partial\theta=0)$ about the z axis. For example,
the mean equilibrium velocity of component j $(j=e,i)$ can be expressed
as $\underset{\sim}{V}_j^0(\underset{\sim}{x}) = V_{j\theta}^0(r)\hat{e}_\theta$, where $\hat{e}_\theta$ is a unit vector in the θ direction, and
$\partial V_{j\theta}^0/\partial\theta = 0 = \partial V_{j\theta}^0/\partial z$.

(b) For present purposes, the electrons and ions are treated as
cold $(T_j \rightarrow 0)$ fluids immersed in the uniform axial magnetic field $B_0\hat{e}_z$.
Within the context of the electrostatic approximation $(\underset{\sim}{B} \simeq B_0\hat{e}_z$ and
$\underset{\sim}{\nabla}\times\underset{\sim}{E} \simeq 0)$, the equation of motion and continuity equation for each plasma
component $(j=e,i)$ can be expressed as

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{V}_j \cdot \nabla\right)\underset{\sim}{V}_j = \frac{e_j}{m_j}\left(\underset{\sim}{E} + \frac{\underset{\sim}{V}_j \times B_0 \hat{e}_z}{c}\right) , \qquad (1)$$

$$\frac{\partial}{\partial t} n_j + \nabla \cdot (n_j \underset{\sim}{V}_j) = 0 , \qquad (2)$$

where $\underset{\sim}{E}(\underset{\sim}{x},t) = -\nabla\phi(\underset{\sim}{x},t)$ is the electric field, $n_j(\underset{\sim}{x},t)$ is the density, $\underset{\sim}{V}_j(\underset{\sim}{x},t)$ is the mean velocity, and e_j and m_j are the charge and mass, respectively, of a particle of species j. In Eq. (1), the spatial variation in B_0 is neglected (low-beta approximation), and the electrostatic potential $\phi(\underset{\sim}{x},t)$ is determined self-consistently from Poisson's equation

$$\nabla^2\phi = 4\pi e(n_e - n_i) , \qquad (3)$$

where $-e$ is the electron charge.

(c) In the stability analysis, flute perturbations with $\partial/\partial z = 0$ are considered. All quantities are expressed as an axisymmetric equilibrium value plus a perturbation, i.e., $\psi(\underset{\sim}{x},t) = \psi_0(r) + \delta\psi(\underset{\sim}{x},t)$. Assuming perturbations with azimuthal harmonic number ℓ, the quantity $\delta\psi(\underset{\sim}{x},t)$ is then expressed as

$$\delta\psi(\underset{\sim}{x},t) = \delta\hat{\psi}_\ell(r)\exp[i(\ell\theta - \omega t)] , \qquad (4)$$

where ω is the complex eigenfrequency.

2.B General Equilibrium Properties

An equilibrium analysis of Eqs. (1)-(3) for general steady-state ($\partial/\partial t = 0$) profiles, $n_j^0(r)$, $V_{j\theta}^0(r)$ and $\phi^0(r)$, proceeds in the following manner. First, it is straightforward to show from Eq. (2) that the functional form of the density profile $n_j^0(r)$ can be specified **arbitrarily**. Moreover, from Eq. (1), equilibrium force balance in the

radial direction can be expressed as $-m_j V_{j\theta}^{02}(r)/r = e_j E_r^0(r) + e_j V_{j\theta}^0(r) B_0/c$, or equivalently

$$\omega_j^2 + \varepsilon_j \omega_{cj} \omega_j - \varepsilon_j \omega_E \omega_{cj} = 0 \quad , \tag{5}$$

where $\varepsilon_j = \text{sgn} e_j$, $\omega_{cj} = |e_j| B_0/m_j c$ is the cyclotron frequency, $\omega_j(r) = V_{j\theta}^0(r)/r$ is the angular velocity of a fluid element, and $\omega_E(r)$ is the angular $\underset{\sim}{E}^0 \times \underset{\sim}{B}_0$ frequency defined by

$$\omega_E = -\frac{c}{rB_0} E_r^0 = \frac{c}{rB_0} \frac{\partial}{\partial r} \phi^0 \quad . \tag{6}$$

Finally, the equilibrium electrostatic potential $\phi^0(r)$ is determined self-consistently from the steady-state Poisson equation

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \phi^0 = 4\pi e(n_e^0 - n_i^0) \quad , \tag{7}$$

where $n_e^0(r)$ and $n_i^0(r)$ are the equilibrium electron and ion density profiles. We note from Eq. (5) that there are two allowed equilibrium values of ω_j. Solving Eq. (5) for ω_j gives

$$\omega_j = \omega_j^\pm \equiv -\frac{\varepsilon_j \omega_{cj}}{2} \left[1 \pm \left(1 + 4\varepsilon_j \frac{\omega_E}{\omega_{cj}} \right)^{1/2} \right] \quad , \tag{8}$$

where the upper sign $(\omega_j = \omega_j^+)$ corresponds to a "fast" rotational equilibrium, and the lower sign $(\omega_j = \omega_j^-)$ corresponds to a "slow" rotational equilibrium. Evidently, in the limit of equilibrium charge neutrality with $\omega_E = 0$, it follows from Eq. (5) that $\omega_j^+ = -\varepsilon_j \omega_{cj}$ and $\omega_j^- = 0$.

3. ELECTROSTATIC STABILITY PROPERTIES

3.A General Eigenvalue Equation

To investigate electrostatic stability behavior, we make use of Eq. (4) and linearize Eqs. (1)–(3) about the axisymmetric equilibrium state characterized by $n_j^0(r)$, $\underset{\sim}{V}_j^0(\underset{\sim}{x})=\omega_j(r)r\hat{\underset{\sim}{e}}_\theta$, and $\underset{\sim}{E}^0(\underset{\sim}{x})=-(\partial\phi^0/\partial r)\hat{\underset{\sim}{e}}_r$. This readily gives

$$-i(\omega-\ell\omega_j)\delta\hat{V}_{jr}-(\varepsilon_j\omega_{cj}+2\omega_j)\delta\hat{V}_{j\theta} = -\frac{e_j}{m_j}\frac{\partial}{\partial r}\delta\hat{\phi} \ , \tag{9}$$

$$-i(\omega-\ell\omega_j)\delta\hat{V}_{j\theta}+[\varepsilon_j\omega_{cj}+\frac{1}{r}\frac{\partial}{\partial r}(r^2\omega_j)]\delta\hat{V}_{jr} = -\frac{e_j}{m_j}\frac{i\ell\delta\hat{\phi}}{r} \ , \tag{10}$$

$$-i(\omega-\ell\omega_j)\delta\hat{n}_j + \frac{1}{r}\frac{\partial}{\partial r}(rn_j^0\delta\hat{V}_{jr}) + \frac{i\ell n_j^0\delta V_{j\theta}^\ell}{r} = 0 \ , \tag{11}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\delta\hat{\phi} - \frac{\ell^2}{r^2}\delta\hat{\phi} = -4\pi\sum_j e_j\delta\hat{n}_j \ , \tag{12}$$

where the index ℓ has been suppressed on the perturbation amplitudes $\delta\hat{V}_{jr}(r)$, $\delta\hat{\phi}(r)$, etc. It is evident from Eqs. (9)–(11) that the perturbations in density and mean fluid velocities can be expressed directly in terms of $\delta\hat{\phi}(r)$. After some straightforward algebra, Eq. (12) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(1-\sum_j\frac{\omega_{pj}^2}{\nu_j^2}\right)\frac{\partial}{\partial r}\delta\hat{\phi}\right]$$

$$-\frac{\ell^2}{r^2}\left(1-\sum_j\frac{\omega_{pj}^2}{\nu_j^2}\right)\delta\hat{\phi} \tag{13}$$

$$= -\frac{\ell\delta\hat{\phi}}{r}\sum_j\frac{1}{\omega-\ell\omega_j}\frac{\partial}{\partial r}\left(\frac{\omega_{pj}^2}{\nu_j^2}(\varepsilon_j\omega_{cj}+2\omega_j)\right) \ ,$$

where $\omega_{pj}^2(r)=4\pi n_j^0(r)e_j^2/m_j$, and $\nu_j^2(r)$ is defined by

$$\nu_j^2(r)=(\omega-\ell\omega_j)^2-(\varepsilon_j\omega_{cj}+2\omega_j)\left[\varepsilon_j\omega_{cj}+\frac{1}{r}\frac{\partial}{\partial r}(r^2\omega_j)\right] \ . \tag{14}$$

The eigenvalue equation (13) can be used to calculate the eigenfunction

$\delta\hat{\phi}(r)$ and complex eigenfrequency ω for a broad class of equilibrium profiles $n_j^0(r)$ and $\omega_j(r)$ consistent with Eqs. (5)-(8).

3.B Eigenvalue Equation for Sharp-Boundary Equilibrium

For present purposes, we specialize to the case where the electron and ion density profiles are rectangular [Fig. 2], i.e.,

$$n_e^0(r) = \begin{cases} n_0=\text{const.}, & 0<r<R_p \ , \\ \\ 0 & , \quad R_p<r<R_c \ , \end{cases} \tag{15}$$

and

$$n_i^0(r) = \begin{cases} fn_0=\text{const.}, & 0<r<R_p \ , \\ \\ 0 & , \quad R_p<r<R_c \ , \end{cases} \tag{16}$$

where f=const.=fractional charge neutralization, and $r=R_c$ is the radial location of a grounded conducting wall. From Eqs. (7), (15) and (16), we find $\phi^0(r)=\pi e n_0(1-f)r^2$ inside the plasma column $(0<r<R_p)$. Making use of Eq. (6), the equilibrium $\underset{\sim}{E}^0\times\underset{\sim}{B}_0$ frequency can be expressed as

$$\omega_E = \frac{2\pi e n_0 c}{B_0} (1-f) = \text{const.} \tag{17}$$

Substituting Eq. (17) into Eq. (8) gives

$$\omega_e = \omega_e^{\pm} = \frac{\omega_{ce}}{2} \left[1 \pm \left(1 - \frac{2\hat{\omega}_{pe}^2}{\omega_{ce}^2} (1-f) \right)^{1/2} \right] , \tag{18}$$

and

$$\omega_i = \omega_i^{\pm} = - \frac{\omega_{ci}}{2} \left[1 \pm \left(1 + \frac{m_i}{m_e} \frac{2\hat{\omega}_{pe}^2}{\omega_{ce}^2} (1-f) \right)^{1/2} \right] , \tag{19}$$

where $\hat{\omega}_{pe}^2=4\pi n_0 e^2/m_e$. It is evident from Eqs. (18) and (19) that the angular velocity profiles ω_e and ω_i are uniform (independent of r) for the electron and ion density profiles illustrated in Fig. 2.

For the rectangular density profiles prescribed by Eqs. (15) and (16), the eigenvalue equation (13) can be expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(1-\sum_j\frac{\omega_{pj}^2}{\nu_j^2}\right)\frac{\partial}{\partial r}\delta\hat{\phi}\right]-\frac{\ell^2}{r^2}\left(1-\sum_j\frac{\omega_{pj}^2}{\nu_j^2}\right)\delta\hat{\phi}$$

$$=\frac{\ell\delta\hat{\phi}}{r}\sum_j\frac{\hat{\omega}_{pj}^2}{\nu_j^2}\frac{\epsilon_j\omega_{cj}+2\omega_j}{\omega-\ell\omega_j}\delta(r-R_p)$$

(20)

where

$$\nu_j^2(r)=(\omega-\ell\omega_j)^2-(\epsilon_j\omega_{cj}+2\omega_j)^2=\text{const.}$$

(21)

and

$$\omega_{pj}^2(r)=\begin{cases}\hat{\omega}_{pj}^2=\text{const.,} & 0<r<R_p\ ,\\[2ex]0 & ,\ \ R_p<r<R_c\ .\end{cases}$$

(22)

In Eqs. (20) and (22), $\hat{\omega}_{pe}^2=4\pi n_0 e^2/m_e$ and $\hat{\omega}_{pi}^2=4\pi fn_0 e^2/m_i$, and use has been made of $\partial\omega_{pj}^2/\partial r=-\hat{\omega}_{pj}^2\delta(r-R_p)$.

3.C Dispersion Relation for Sharp-Boundary Equilibrium

Except at the point $r=R_p$, it is clear from Eqs. (20)-(22) that $\delta\hat{\phi}(r)$ satisfies the vacuum Poisson equation $r^{-1}(\partial/\partial r)[r\partial\delta\hat{\phi}/\partial r]-(\ell^2/r^2)\delta\hat{\phi}=0$. Therefore, the solution to Eq. (20) can be expressed as

$$\delta\hat{\phi}_i(r)=A(r/R_p)^\ell\ ,\quad 0\leq r<R_p\ ,$$

(23)

inside the plasma column, and

$$\delta\hat{\phi}_o(r)=A\left(\frac{r}{R_p}\right)^\ell\frac{(1-R_c^{2\ell}/r^{2\ell})}{(1-R_c^{2\ell}/R_p^{2\ell})}\ ,\quad R_p<r\leq R_c\ ,$$

(24)

in the vacuum region between the surface of the plasma column and the conducting wall. Note that the perturbed potential is continuous at the surface of the plasma column with $\delta\hat{\phi}_i(r=R_p)=\delta\hat{\phi}_o(r=R_p)=A$. Moreover, the perturbed potential vanishes at the conducting wall,

i.e., $\delta\hat{\phi}_o(r=R_c)=0$.

The dispersion relation that determines the complex eigenfrequency ω is obtained by multiplying Eq. (20) by r and integrating from $R_p(1-\varepsilon)$ to $R_p(1+\varepsilon)$, with $\varepsilon\to0_+$. This gives

$$R_p\left(\frac{\partial}{\partial r}\,\delta\hat{\phi}_o\right)_{r=R_p} - R_p\left(1 - \sum_j \frac{\hat{\omega}_{pj}^2}{\nu_j^2}\right)\left(\frac{\partial}{\partial r}\,\delta\hat{\phi}_i\right)_{r=R_p}$$
$$= \ell\delta\hat{\phi}(r=R_p)\sum_j \frac{\hat{\omega}_{pj}^2}{\nu_j^2}\,\frac{\varepsilon_j\omega_{cj}+2\omega_j}{\omega-\ell\omega_j}\,. \tag{25}$$

Substituting Eqs. (23) and (24) into Eq. (25) and rearranging terms, we obtain

$$0 = \frac{1}{1-(R_p/R_c)^{2\ell}} - \sum_j \frac{\hat{\omega}_{pj}^2}{2(\omega-\ell\omega_j)[(\omega-\ell\omega_j)+(\varepsilon_j\omega_{cj}+2\omega_j)]}\,. \tag{26}$$

Throughout the remainder of this article, we assume that the electron and ion equilibria are rotating in the "slow" rotational mode with

$$\omega_e=\omega_e^- \,, \quad \text{and} \quad \omega_i=\omega_i^- \,. \tag{27}$$

In this case, Eq. (26) can be expressed as

$$0 = \frac{1}{1-(R_p/R_c)^{2\ell}} - \frac{\hat{\omega}_{pe}^2}{2(\omega-\ell\omega_e^-)[(\omega-\ell\omega_e^-)-(\omega_e^+-\omega_e^-)]}$$
$$- \frac{\hat{\omega}_{pi}^2}{2(\omega-\ell\omega_i^-)[(\omega-\ell\omega_i^-)-(\omega_i^+-\omega_i^-)]}\,, \tag{28}$$

where $\omega_e^\pm$ and $\omega_i^\pm$ are defined in Eqs. (18) and (19). The linear dispersion relation (28) can be used to investigate electrostatic stability properties for a broad range of plasma parameters f, $\hat{\omega}_{pe}^2/\omega_{ce}^2$, etc.

As a point of reference, we note that the ion resonance instability has previously been studied in the low-density, low-frequency regime

characterized by[11,12]

$$\frac{2\hat{\omega}_{pe}^2}{\omega_{ce}^2}(1-f) \ll 1 \ ,$$

(29)

$$|\omega - \ell\omega_e^-| \ll |\omega_e^+ - \omega_e^-| \ .$$

Within the context of the inequalities in Eq. (29), the linear dispersion relation (28) can be approximated by

$$0 = \frac{1}{1-(R_p/R_c)^{2\ell}} + \frac{\hat{\omega}_{pe}^2}{2\omega_{ce}(\omega-\ell\omega_E)}$$
$$- \frac{\hat{\omega}_{pi}^2}{2(\omega-\ell\omega_i^-)[(\omega-\ell\omega_i^-)-(\omega_i^+-\omega_i^-)]} \ ,$$

(30)

where use has been made of $\omega_e^- \approx \omega_E$ and $\omega_e^+ - \omega_e^- \approx \omega_{ce}$ [Eqs. (17), (18) and (29)]. Equation (30) is identical to the dispersion relation obtained by Levy et al.[11] within the framework of a fluid model in which the electron motion is approximated by $\underset{\sim}{V}_e = c(\underset{\sim}{E} \times \hat{\underset{\sim}{e}}_z)/B_0$. Throughout the remainder of this article, Eq. (30) is referred to as the "reference dispersion relation" (RDR). In order to assess the influence of strong self electric fields on stability behavior $[2(\hat{\omega}_{pe}^2/\omega_{ce}^2)(1-f) \leq 1]$, it will be useful, from time to time, to compare the stability information obtained from the unexpanded dispersion relation (28) with that obtained from Eq. (30). In this regard, we emphasize that the inequalities in Eq. (29) pose a particularly severe limitation on the range of validity of Eq. (30). The present paper is intended to alleviate this situation by a thorough investigation of the unexpanded dispersion relation (28).

3.D Dimensionless Parameters

It is evident from Eqs. (17)-(19) and the linear dispersion relation (28) that the four dimensionless parameters

$$\frac{\hat{\omega}_{pe}^2}{\omega_{ce}^2} \; , \quad f \; , \quad \frac{m_e}{m_i} \; , \quad \frac{R_p}{R_c} \; , \tag{31}$$

can be used to characterize the plasma equilibrium. Moreover, an important dimensionless parameter that measures the strength of the equilibrium electric force on an electron fluid element is [Eq. (8)]

$$\delta \equiv 4 \, \frac{\omega_E}{\omega_{ce}} = \frac{2\hat{\omega}_{pe}^2}{\omega_{ce}^2} \, (1-f) \; , \tag{32}$$

which can be constructed from the dimensionless parameters in Eq. (31). Because

$$\frac{2\hat{\omega}_{pe}^2}{\omega_{ce}^2} \, (1-f) \leq 1 \tag{33}$$

is necessarily required for existence of the electron equilibrium [Eq. (18)], we note from Eqs. (32) and (33) that δ can span the range $0 \leq \delta \leq 1$.

For future convenient reference, in Fig. 3 are shown plots of ω_E/ω_{ce} [Eq. (17)], $\omega_e^{\pm}/\omega_{ce}$ [Eq. (18)] and $\omega_i^{\pm}/\omega_{ci}$ [Eq. (19)] versus the dimensionless parameter $\delta = (2\hat{\omega}_{pe}^2/\omega_{ce}^2)(1-f)$ for $m_e/m_i = 1/1836$. In the limit of maximum charge density for existence of the equilibrium ($\delta = 1$), we note from Fig. 3 that $\omega_i^{\pm}/\omega_{ci} \approx \mp(m_i/4m_e)^{1/2}$ and $\omega_e^{\pm}/\omega_{ce} = 0.5$. Finally, in Fig. 4, we illustrate the region of the parameter space $(f, \hat{\omega}_{pe}^2/\omega_{ce}^2)$ corresponding to existence of the equilibrium [Eq. (33)]. Note that the equilibrium electron density can exceed the Brillouin flow limit ($\hat{\omega}_{pe}^2/\omega_{ce}^2 = 0.5$) provided there is sufficient

charge neutralization that $f \geq 1 - 0.5(\omega_{ce}^2/\hat{\omega}_{pe}^2)$.

4. STABILITY ANALYSIS

The growth rate $\gamma = \text{Im}\omega$ and real oscillation frequency $\omega_r = \text{Re}\omega$ have been obtained numerically from Eq. (28) for a broad range of plasma parameters f, $\hat{\omega}_{pe}^2/\omega_{ce}^2$ and R_p/R_c. In this section, we summarize the essential features of these stability studies. The analysis is restricted to nonneutral proton-electron plasmas ($m_i/m_e = 1836$), and the growth rate and real frequency are measured in units of the lower hybrid frequency

$$\omega_{LH} = (\omega_{ce}\omega_{ci})^{1/2} .$$

In order to illustrate the need for an improved dispersion relation at large beam densities, Fig. 5 shows a plot of the normalized growth rate γ/ω_{LH} and oscillation frequency ω_r/ω_{LH} versus $\hat{\omega}_{pe}^2/\omega_{ce}^2$ obtained from Eq. (28) (the unexpanded dispersion relation) and Eq. (30) (the reference dispersion relation), for $\ell=2$, $R_p/R_c = 0.5$ and $f = 0.45$. Note that ω_r/ω_{LH} is plotted only for the range of $\hat{\omega}_{pe}^2/\omega_{ce}^2$ corresponding to instability ($\gamma > 0$). Moreover, the abscissa in Fig. 5 extends to $\hat{\omega}_{pe}^2/\omega_{ce}^2 = 0.91$, since physically allowed equilibria exist for $\hat{\omega}_{pe}^2/\omega_{ce}^2 \leq 0.5/(1-0.45) = 0.91$ [Eq. (33) and Fig. 4]. Several points are noteworthy in Fig. 5. First, the unexpanded dispersion relation (28) predicts instability for $0 < \hat{\omega}_{pe}^2/\omega_{ce}^2 < 0.34$, whereas the reference dispersion relation (30) predicts instability for the entire range of allowed equilibrium density, $0 < \hat{\omega}_{pe}^2/\omega_{ce}^2 < 0.91$. Second, the maximum growth rate can be a substantial fraction of ω_{LH}. In particular, for the parameters assumed in Fig. 5, the maximum

growth rate obtained from Eq. (28) is $\gamma_{MAX}=0.36\ \omega_{LH}$, which occurs

for $\hat{\omega}_{pe}^2/\omega_{ce}^2=0.24$. Finally, it is evident from Fig. 5 that the reference

dispersion relation (30) breaks down at rather modest values of beam

density ($\hat{\omega}_{pe}^2/\omega_{ce}^2\gtrsim0.1$), thereby underlining the need for the improved

dispersion relation in Eq. (28). Throughout the remainder of this

section, the stability analysis is based entirely on Eq. (28).

Stability boundaries in the parameter space $(f,\ \hat{\omega}_{pe}^2/\omega_{ce}^2)$ are

illustrated in Figs. 6 and 7. In Fig. 6, the solid curves correspond

to the stability boundaries ($\gamma=0$) obtained from Eq. (28) for $\ell=1$,

and several values of R_p/R_c. For a given value of R_p/R_c, the region

of $(f,\ \hat{\omega}_{pe}^2/\omega_{ce}^2)$ parameter space <u>above</u> the curve corresponds to

instability ($\gamma>0$), whereas the region of parameter space <u>below</u> the

curve corresponds to stability ($\gamma=0$). For low beam densities

(small values of $\hat{\omega}_{pe}^2/\omega_{ce}^2$), it is evident from Fig. 6 that the

system is stable for $\ell=1$, provided R_p/R_c is sufficiently small.

On the other hand, for high beam densities, wall stabilization

occurs provided R_p/R_c is sufficiently large, i.e., provided the

conducting wall is located sufficiently close to the plasma surface.

In Fig. 7, the solid curves correspond to the stability boundaries

obtained from Eq. (28) for $R_p/R_c=0.5$ and several values of azimuthal

mode number ℓ. For a given density of physical interest ($\hat{\omega}_{pe}^2/\omega_{ce}^2>0.01$,

say), we note that the number of unstable modes increases rapidly

as the fractional charge neutralization f is increased to sufficiently

large values.

The dependence of stability properties on fractional charge

neutralization is further illustrated in Figs. 8 and 9 where the

normalized growth rate γ/ω_{LH} and oscillation frequency ω_r/ω_{LH} are

plotted versus f for $\hat{\omega}_{pe}^2/\omega_{ce}^2=0.01$ (Fig. 8) and $\hat{\omega}_{pe}^2/\omega_{ce}^2=0.5$ (Fig. 9),

and several values of mode number ℓ. Also, $R_p/R_c=0.5$ is assumed in

Figs. 8 and 9. In Figs. 8(b) and 9(b), ω_r/ω_{LH} is plotted only for the

ranges of f corresponding to instability ($\gamma>0$). Note the rapid increase

in the number of unstable modes when f is increased to sufficiently

large values. Several important features are evident from Figs. 8 and

9. First, for the low-density case illustrated in Fig. 8 ($\hat{\omega}_{pe}^2/\omega_{ce}^2=0.01$),

we note that only the fundamental mode ($\ell=1$) is unstable when f<0.13.

This is consistent with the results previously obtained by Levy et al.

for $\hat{\omega}_{pe}^2/\omega_{ce}^2$, f<<1. Second, the maximum growth rate for each mode

number ℓ is a slowly increasing function of f. Evidently, as the

fractional charge neutralization f approaches unity, the number

of unstable modes tends to infinity within the context of the present

cold-fluid model. Moreover, in the limit of equilibrium charge

neutrality (f=1), the self electric field is equal to zero and the

plasma column is stable ($\gamma=0$ in Figs. 8 and 9). Finally, for low

beam densities, we note that the real frequency ω_r exhibits a

nearly linear dependence on fractional charge neutralization f

(Fig. 8). In contrast, when the equilibrium self-electric field

is sufficiently large (Fig. 9), ω_r exhibits a strongly nonlinear

dependence on f.

Shown in Figs. 10 and 11 are plots of normalized growth rate

γ/ω_{LH} and oscillation frequency ω_r/ω_{LH} versus $\hat{\omega}_{pe}^2/\omega_{ce}^2$ for f=0.1

(Fig. 10) and f=0.8 (Fig. 11), and several values of mode number ℓ.

Also, $R_p/R_c=0.5$ is assumed in Figs. 10 and 11. For small values of

fractional charge neutralization, it is evident that the fundamental

mode ($\ell=1$) is the most unstable mode. For example, for the f=0.1 case

shown in Fig. 10, the maximum growth rate ($\gamma_{MAX} \approx 0.093 \ \omega_{LH}$) occurs for $\ell=1$ and $\hat{\omega}_{pe}^2/\omega_{ce}^2 \approx 0.12$. This is in contrast with Fig. 11, where the maximum growth rate for f=0.8 ($\gamma_{MAX} \approx 2.4 \ \omega_{LH}$) occurs for $\ell=3$ and $\hat{\omega}_{pe}^2/\omega_{ce}^2 = 2.5$ (the maximum allowed density for radial confinement). Furthermore, for the f=0.1 case shown in Fig. 10, we note that the ion resonance instability is completely stabilized above some critical value of beam density ($\hat{\omega}_{pe}^2/\omega_{ce}^2 \gtrsim 0.17$). On the other hand, for larger values of fractional charge neutralization (f=0.8 in Fig. 11), it is evident that instability exists for the entire allowed range of $\hat{\omega}_{pe}^2/\omega_{ce}^2$. Moreover, several mode numbers ℓ are unstable for a given value of $\hat{\omega}_{pe}^2/\omega_{ce}^2$, a feature generally characteristic of a nonneutral plasma column with significant charge neutralization (see also Figs. 7-9).

Of considerable interest for experimental application is the stability behavior for specified values of f, $\hat{\omega}_{pe}^2/\omega_{ce}^2$ and R_p/R_c. Typical results are shown in Figs. 12 and 13 where γ/ω_{LH} and ω_r/ω_{LH} are plotted versus mode number ℓ for $\hat{\omega}_{pe}^2/\omega_{ce}^2=0.01$ (Fig. 12) and $\hat{\omega}_{pe}^2/\omega_{ce}^2=0.5$ (Fig. 13), and several values of fractional charge neutralization f. Also, $R_p/R_c=0.5$ is assumed in Figs. 12 and 13, and graphical results are presented only for the unstable mode numbers with $\gamma>0$. For the low-density case in Fig. 12, we note that maximum growth occurs for $\ell=2$ when f=0.2, and for $\ell=12$ when f=0.8. Evidently, for low beam densities, there is a broad spectrum of unstable modes when the fractional charge neutralization is sufficiently large. On the other hand, for higher beam densities, fewer unstable modes are excited. For example, in Fig. 13, maximum growth occurs for $\ell=1$ when f=0.2, and for $\ell=5$ when f=0.8. Comparing Figs. 12 and 13, we also note that the maximum growth rate (measured in units of ω_{LH}) is larger in the high-density case (Fig. 13) than in the low-density

case (Fig. 12).

We conclude this section by emphasizing that stability properties also exhibit a sensitive dependence on the location of the conducting wall. This is illustrated in Fig. 14 where the normalized growth rate γ/ω_{LH} and oscillation frequency ω_r/ω_{LH} are plotted versus R_p/R_c for $\ell=1$, $\hat{\omega}^2_{pe}/\omega^2_{ce}=0.5$, and several values of fractional charge neutralization f. Evidently, the $\ell=1$ mode is stabilized whenever the conducting wall is located sufficiently close to the plasma surface. Moreover, as f is increased, the conducting wall must be situated even closer to the plasma surface to assure stability. A qualitatively similar dependence of stability behavior on R_p/R_c is obtained for higher ℓ values.

5. SUMMARY AND CONCLUSIONS

In this paper, we have examined the influence of strong self-electric fields on the ion resonance instability in a nonneutral plasma column. The analysis was carried out within the framework of a macroscopic cold-fluid model (Section 2), and electrostatic stability properties were investigated in detail for the case of rectangular electron and ion density profiles (Sections 3 and 4). One of the most important conclusions of this study is that the equilibrium self-electric field can have a large influence on stability behavior. In particular, stability properties for $\delta \lesssim 1$ differ substantially from those obtained when $\delta << 1$ (see, for example, Fig. 5). Moreover, for a nonneutral plasma column with significant charge neutralization, it is found that the fundamental mode ($\ell = 1$) is not the most unstable mode (see, for example, Figs. 8 and 9). However, in the special limiting case where $f << 1$ and $\delta << 1$, the $\ell = 1$ mode can have the largest growth rate, which is consistent with the result obtained previously by Levy et al.[12]

Finally, we emphasize that the present analysis is based on a macroscopic cold-fluid model in which the ion (and electron) motion is assumed to be laminar. Although this is a reasonable approximation when $r_{Li} << R_p$ (where r_{Li} is the characteristic thermal ion Larmor radius), we expect significant modifications to the stability behavior when $r_{Li} \sim R_p$. The influence of finite ion Larmor radius effects on the ion resonance instability is currently under investigation[14] within the framework of a hybrid Vlasov-fluid model, which treats the electrons as a macroscopic, cold fluid and the ions in a fully kinetic (Vlasov) manner.

ACKNOWLEDGMENTS

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FIGURE CAPTIONS

Fig. 1   Equilibrium configuration and coordinate system.

Fig. 2   Rectangular density profiles for electrons and ions
[Eqs. (15) and (16)].

Fig. 3   Plots of (a) ω_E/ω_{ce} [Eq. (17)]. (b) $\omega_e^{\pm}/\omega_{ce}$ [Eq. (18)]
and (c) $\omega_i^{\pm}/\omega_{ci}$ [Eq. (19)] versus $\delta=(2\hat{\omega}_{pe}^2/\omega_{ce}^2)(1-f)$ for
$m_i/m_e=1836$.

Fig. 4   The shaded region of the parameter space $(f,\hat{\omega}_{pe}^2/\omega_{ce}^2)$ corres-
ponds to physically allowed equilibria satisfying $f\geq 1-\omega_{ce}^2/2\hat{\omega}_{pe}^2$.

Fig. 5   Plots of (a) growth rate γ and (b) real frequency ω_r versus
$\hat{\omega}_{pe}^2/\omega_{ce}^2$ obtained from Eq. (28) and the reference dispersion
relation (30) for $\ell=2$, $R_p/R_c=0.5$, $f=0.45$ and $m_i/m_e=1836$.

Fig. 6   Stability boundaries [Eq. (28)] in the parameter space
$(f,\hat{\omega}_{pe}^2/\omega_{ce}^2)$ for $\ell=1$, $m_i/m_e=1836$ and $R_p/R_c=0.25$, 0.5, 0.75.

Fig. 7   Stability boundaries [Eq. (28)] in the parameter space
$(f,\hat{\omega}_{pe}^2/\omega_{ce}^2)$ for $R_p/R_c=0.5$, $m_i/m_e=1836$ and several values of ℓ.

Fig. 8   Plots of (a) growth rate γ and (b) real frequency ω_r versus
f [Eq. (28)] for $\hat{\omega}_{pe}^2/\omega_{ce}^2=0.01$, $m_i/m_e=1836$, $R_p/R_c=0.5$ and
several values of ℓ.

Fig. 9   Plots of (a) growth rate γ and (b) real frequency ω_r versus
f [Eq. (28)] for $\hat{\omega}_{pe}^2/\omega_{ce}^2=0.5$, $m_i/m_e=1836$, $R_p/R_c=0.5$ and several
values of ℓ.

Fig. 10   Plots of (a) growth rate γ and (b) real frequency ω_r versus $\hat{\omega}_{pe}^2/\omega_{ce}^2$ [Eq. (28)] for $f=0.1$, $m_i/m_e=1836$, $R_p/R_c=0.5$ and several values of ℓ.

Fig. 11   Plots of (a) growth rate γ and (b) real frequency ω_r versus $\hat{\omega}_{pe}^2/\omega_{ce}^2$ [Eq. (28)] for $f=0.8$, $m_i/m_e=1836$, $R_p/R_c=0.5$ and several values of ℓ.

Fig. 12   Plots of (a) growth rate γ and (b) real frequency ω_r versus ℓ [Eq. (28)] for $\hat{\omega}_{pe}^2/\omega_{ce}^2=0.01$, $m_i/m_e=1836$, $R_p/R_c=0.5$ and several values of f.

Fig. 13   Plots of (a) growth rate γ and (b) real frequency ω_r versus ℓ [Eq. (28)] for $\hat{\omega}_{pe}^2/\omega_{ce}^2=0.5$, $m_i/m_e=1836$, $R_p/R_c=0.5$ and several values of f.

Fig. 14   Plots of (a) growth rate γ and (b) real frequency ω_r versus R_p/R_c [Eq. (28)] for $\hat{\omega}_{pe}^2/\omega_{ce}^2=0.5$, $\ell=1$, $m_i/m_e=1836$, and several values of f.
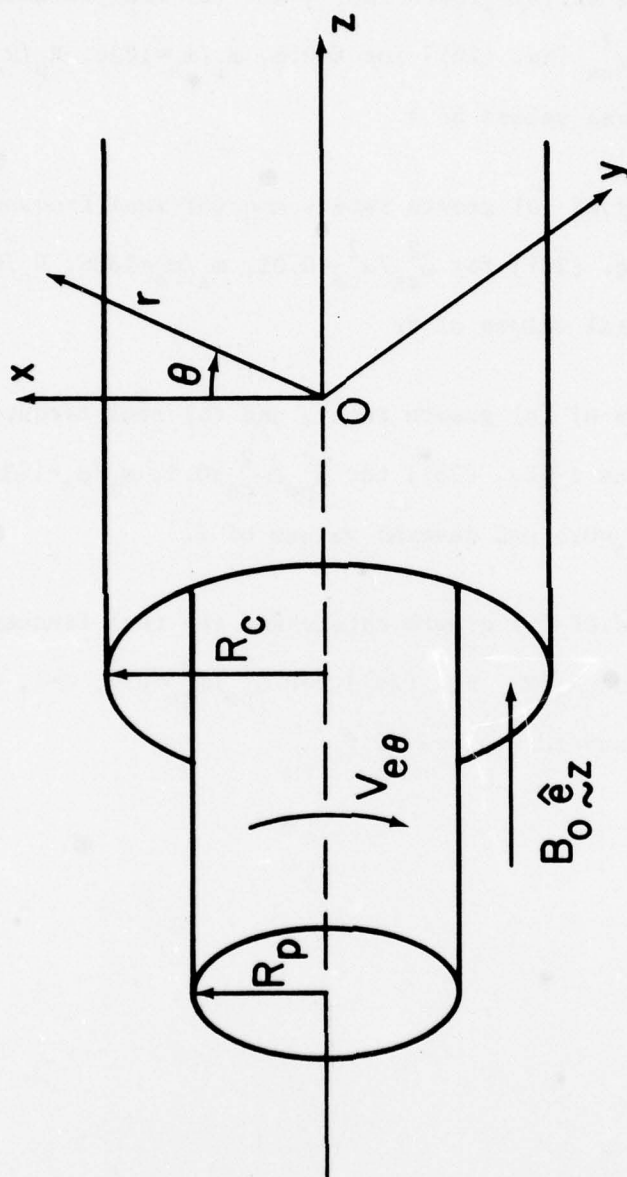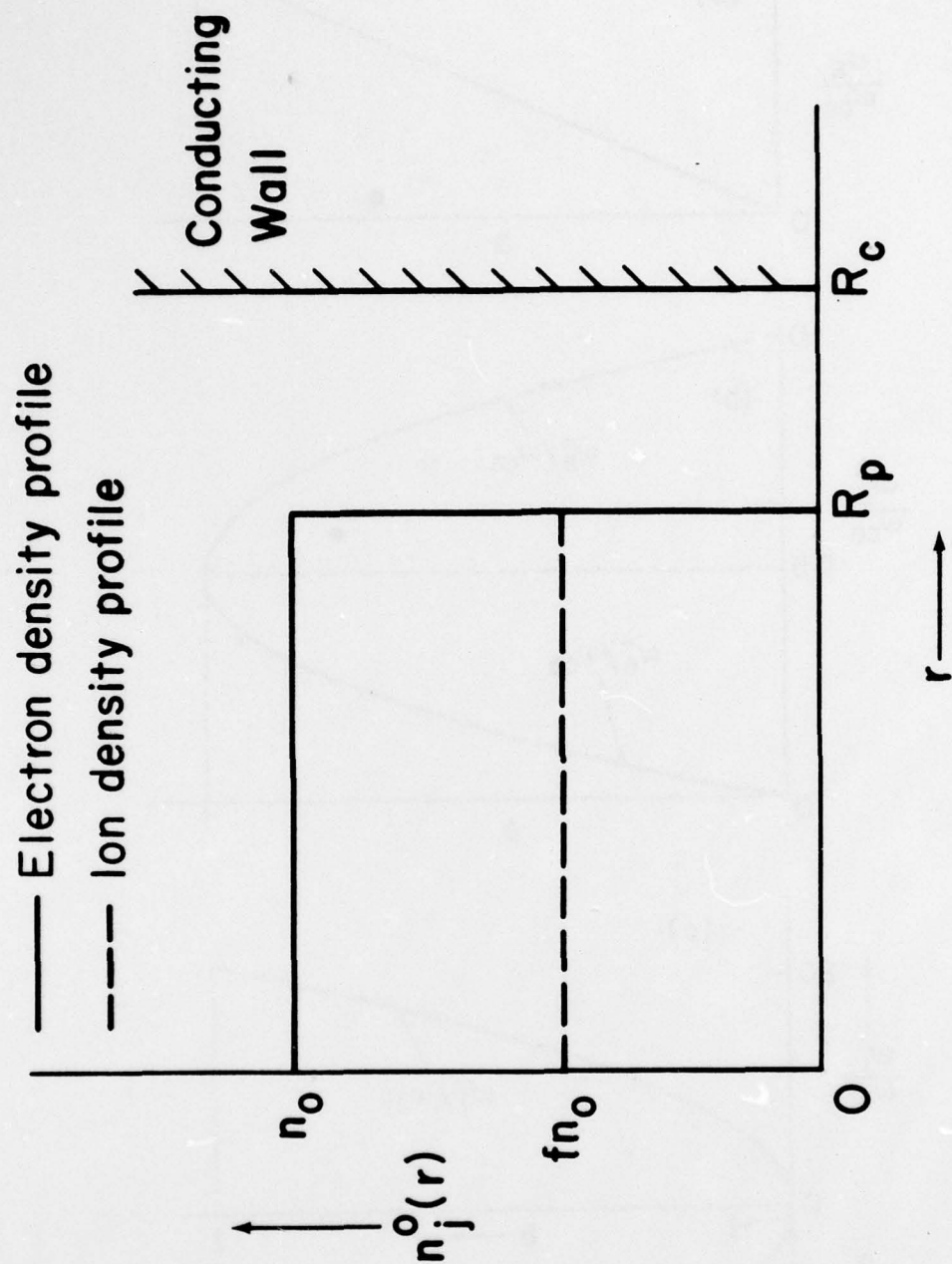
Fig. 1

Fig. 2
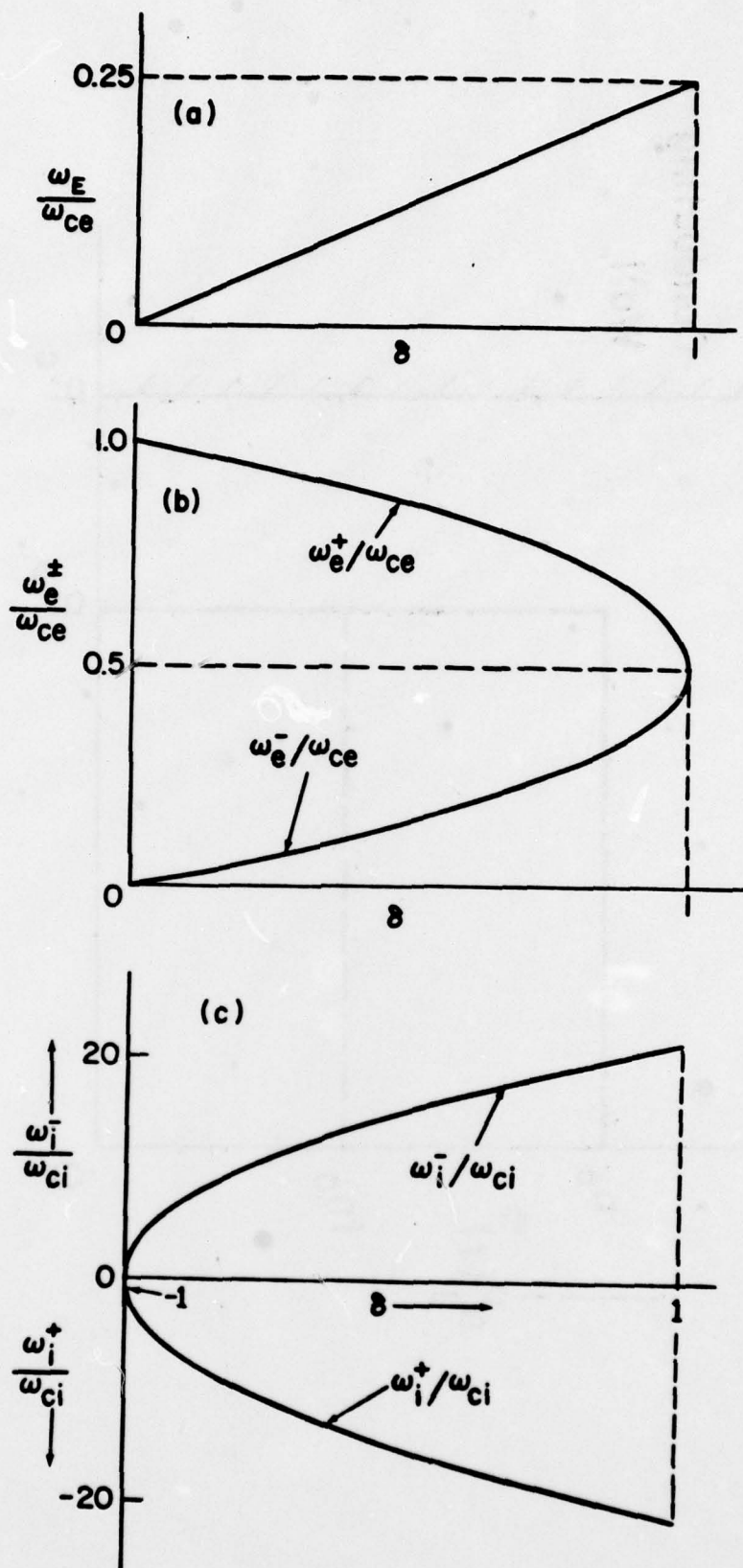
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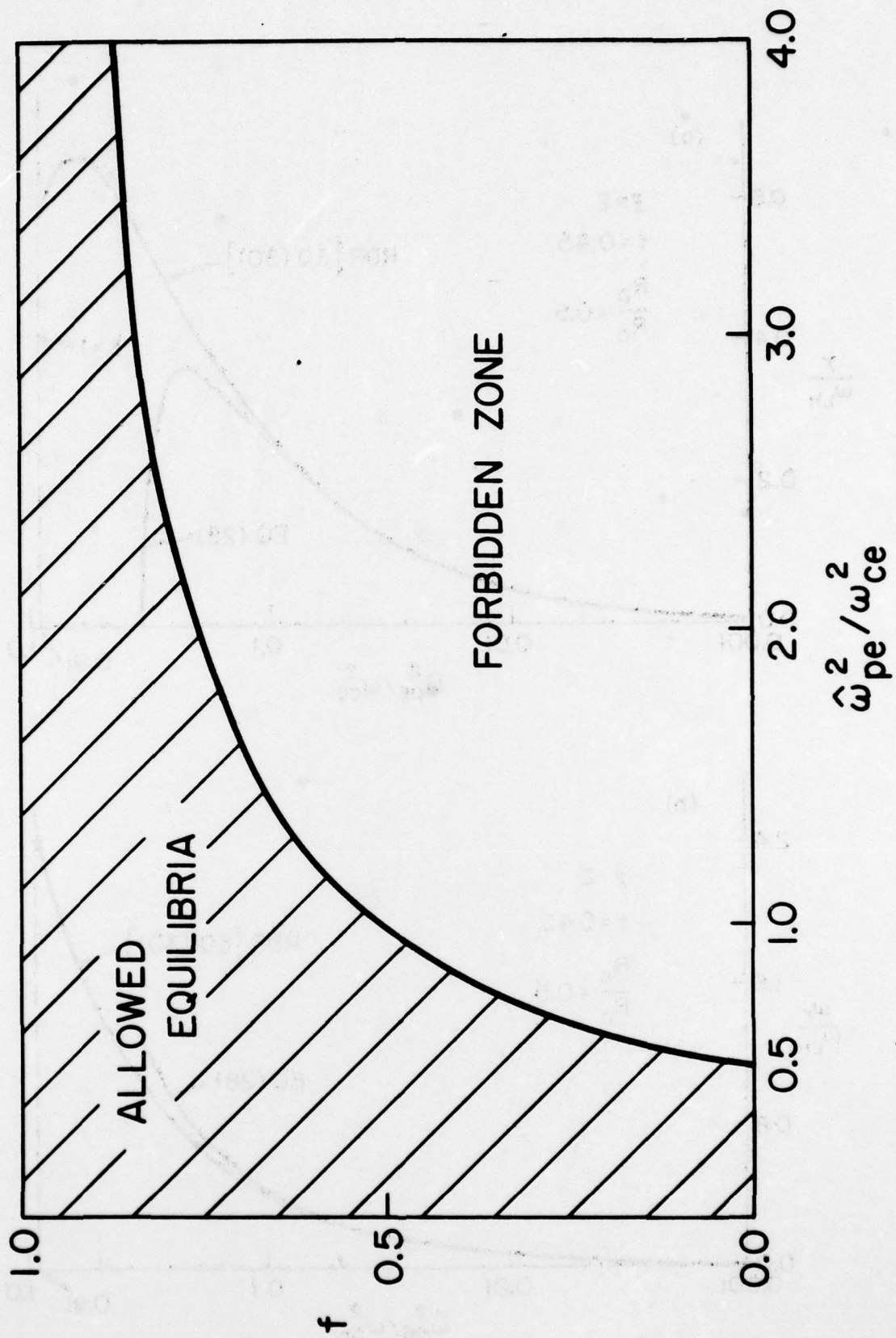
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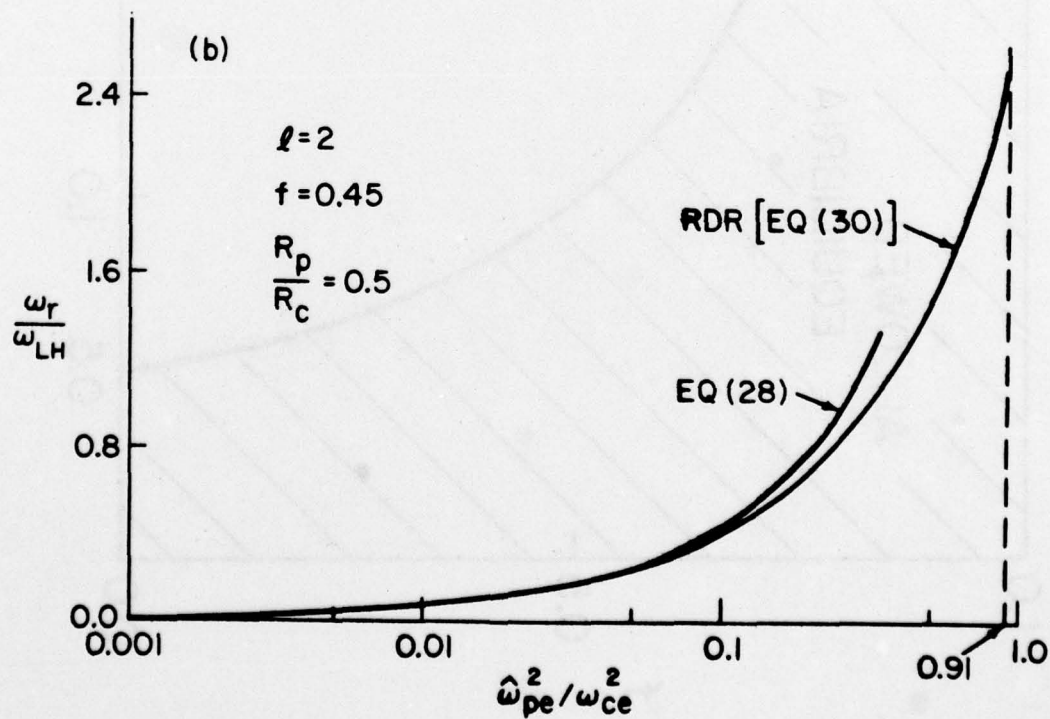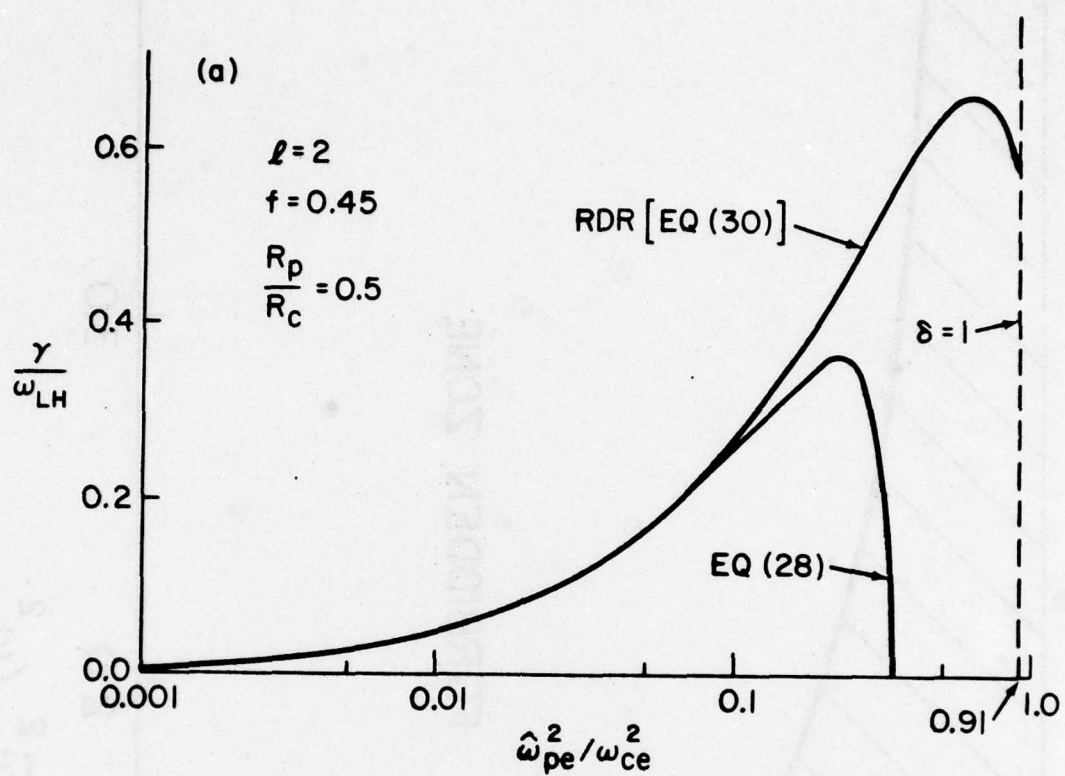
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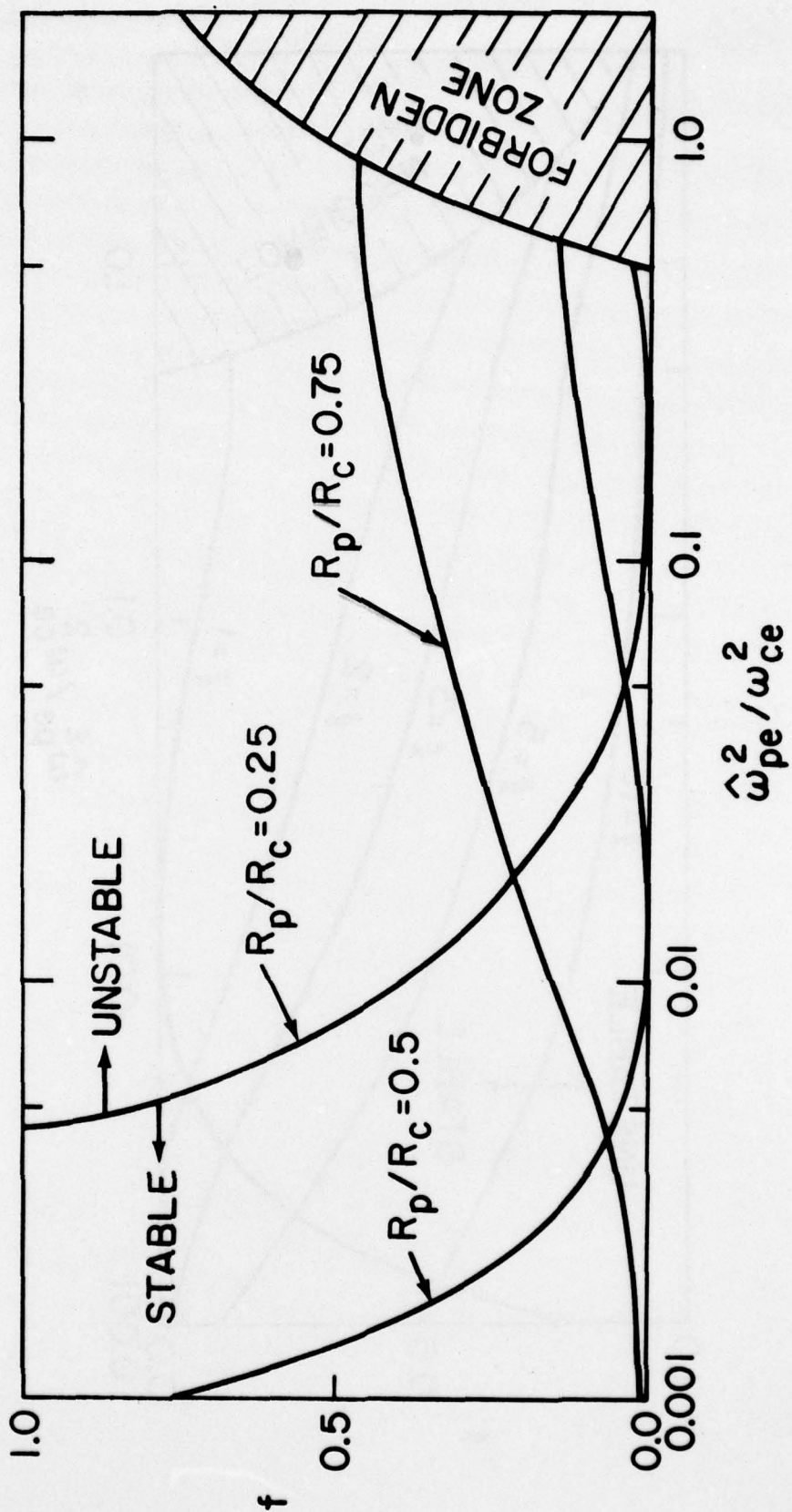
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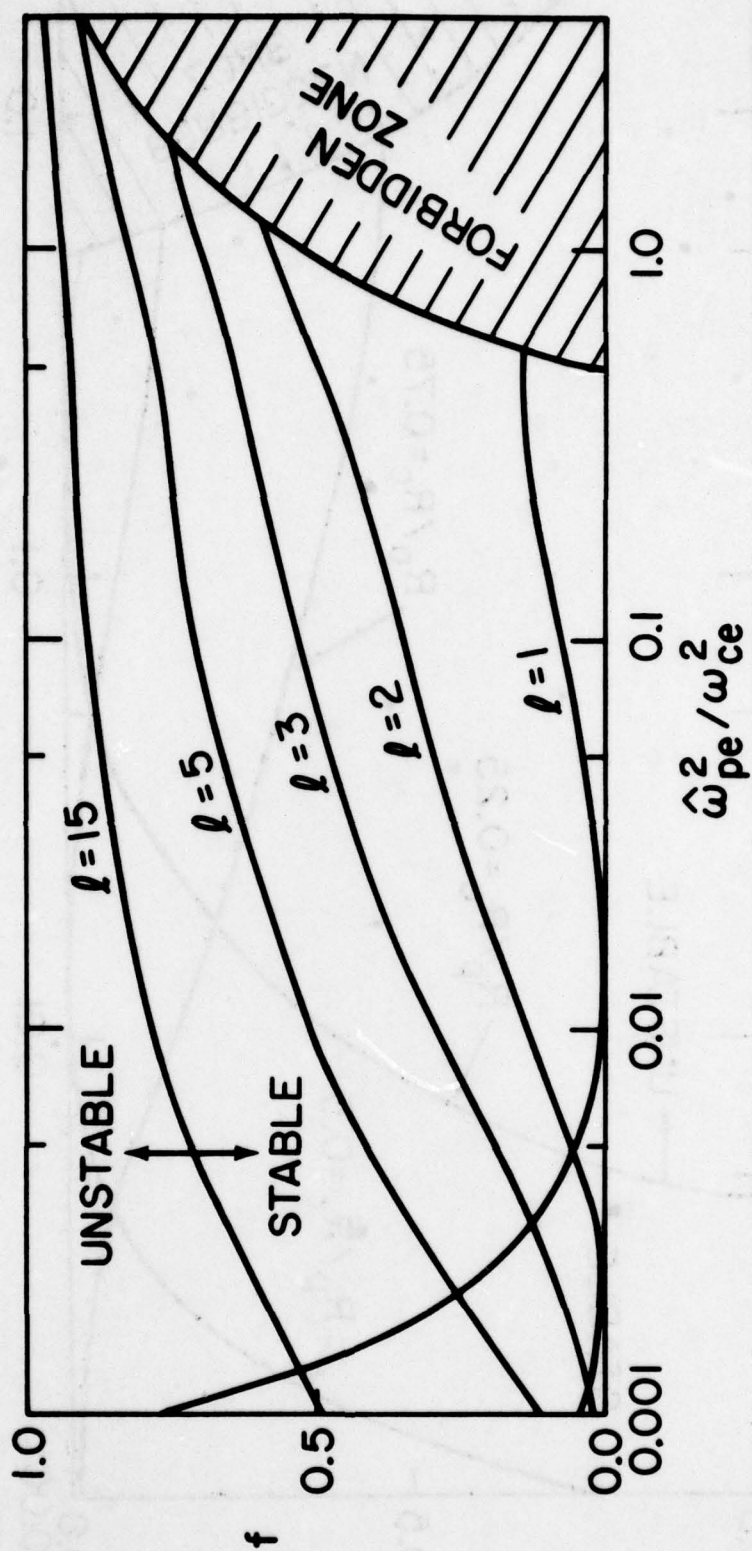
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